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Painlevé singularity analysis of the perfect fluid Bianchi type-IX relativistic cosmological model

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Abstract. We perform the Painlevé test (i.e. the Kowalevskaya–Gambier test or the perturbative test) for the perfect fluid Bianchi type-IX relativistic cosmological model, in order to predict some probable chaotic regimes. This technique enables us to detect every single-valued particular solution associated with a given relativistic dynamical system. Several physically interesting cases are studied. These include radiation-dominated and matter-dominated universes, the case of a cosmological constant and the case of the so-called stiff-matter. In all cases but the cosmological constant case, the models studied do not pass the Painlevé test, exhibit infinitely many movable logarithms and are therefore probably chaotic. Moreover, we show that some physically interesting single-valued particular solutions present in the vacuum case cannot subsist for a perfect fluid cosmological model. In particular, in the radiation and dust cases, we show that there cannot exist any exact and closed-form axisymmetric solution.

1. Introduction

Classical general relativity dramatically predicts, under rather general conditions, the occurrence of spacetime singularities. In the early 1970s, Belinskij, Khalatnikov and Lifchitz (hereafter BKL) discovered a 'general solution' to the gravitational field equations, in the vicinity of the initial singularity [1]. Although this research has been carried out in a quite general inhomogeneous geometrical framework, the most interesting features of this 'general solution' can indeed be understood within the much more simple framework of the so-called 'mixmaster' universe extensively studied, mainly by BKL, Barrow [2] and Misner [3].

This 'mixmaster' model is a homogeneous diagonal cosmological model whose homogeneity group is $SO(3)$. The oscillatory mode of approach to the singular point has been studied by BKL with the help of some approximation techniques, gathering strong evidence which favours a deeply chaotic (ergodic and strongly mixing) behaviour for the underlying nonlinear dynamical ordinary differential system. In the late 1960s, Misner developed a Hamiltonian formulation of this problem and thus produced qualitative results perfectly compatible with the discrete dynamics described by BKL. Both techniques show that the dynamics of this model is correlated to a discrete transformation whose metric properties were considered by Barrow and which is known to be chaotic, exhibiting the Bernoullian property.

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However, a deep controversy soon developed about the real significance of such results, and the debate strongly emphasized the need for an exact (as opposed to approximate) treatment of the differential system governing the fate of the model in the vicinity of the initial cosmological singularity. Numerous numerical investigations have been carried out, within various different frameworks and using a vast number of techniques [4]. However, the reason for the ambiguous results thus obtained is the crucial dependence of the classical criteria of chaos with respect to the temporal gauge used to write down the metric tensor and is thus related to the use of classical (i.e. Newtonian) criteria of chaos within a general covariant theoretical framework (i.e. Einsteinian relativity). Another related problem is the non-compactness of the system's phase space, arising from the fact that the metric functions and their time derivatives are not bounded in principle in the neighbourhood of a cosmological singularity.

The non-compactness problem has led Bogoyavlenskij and his co-workers to the study of this dynamics in a framework such that the physical region in the phase space becomes a compact manifold [5]. One of the most important results of this study is the fact that almost every trajectory of the dynamical system (during the contraction of space) is such that its dynamics can be approximately described in terms of separatrix sequences. So studied, this dynamics possesses a (strange) attractor which is not a smooth (i.e. $\mathcal{C}^\infty$) manifold. This is clearly a very strong indication of the non-integrability of the differential system under study, in the class of smooth solutions. This indicates a probably chaotic oscillatory behaviour of the underlying differential system. Using some numerical relativity methods, Demaret and De Rop [4] have shown the fractal nature of the power spectrum associated with numerical solutions of this cosmological dynamics and, thus, the probably chaotic behaviour of the 'mixmaster' model in Misner–Chitre coordinates, where the non-compactness problem vanishes.

Our treatment of this dynamics should be exact and analytical in order to avoid the problems and ambiguousness stated above. Recently, such an analytical treatment has been proposed and undertaken by Contopoulos, Grammaticos and Ramani (hereafter CGR) and by Latifi, Musette and Conte (hereafter LMC), both in the context of a vacuum model [6]. Both these works investigate the integrability (in the Painlevé sense, to be recalled later) of the 'mixmaster' model by an analysis of the singularities exhibited by the most general solution of the underlying differential system. The method used is known as the 'Painlevé method' and can lead one to gain strong indications about the presumably chaotic behaviour of the model under study. We recall that the aim of this Painlevé test is not to provide the solution in an exact and closed form, but to generate necessary conditions (generally not sufficient) for integrability in the Painlevé sense.

Here we shall stress the fact that this Painlevé method is merely a tool very widely used during the quest for integrable systems. This notion of 'integration' is clearly a delicate one, and the behaviour of the general solution of differential systems in the complex plane (i.e. the singularity structure of this general solution) turns out to be the most fundamental factor while discussing this issue. The practical level of integrability is the integrability in the Poincaré sense. To integrate a differential system in the sense of Poincaré is to find, for the general solution, a finite expression (possibly multi-valued) in a finite number of functions (by definition, a function is a single-valued application of the Riemann sphere onto itself). On the other hand, new functions can be defined by means of nonlinear ordinary differential equations (ODEs). Since the singularities of the solutions of nonlinear ODEs may be movable in the complex plane, one must consider one and only one fundamental issue: does the general solution of the ODE under study exhibit movable critical singularities or not? In the latter case, the ODE is said to be integrable in the Painlevé sense, and this is clearly the most elementary level of integrability.

Necessary conditions for an ODE to be integrable in the Painlevé sense are produced by the Painlevé test, which is based on two fundamental theorems (the existence theorem of Cauchy and Picard and the perturbation theorem of Poincaré and Lyapunov). Both theorems express a local property (and thus can only be used to prove non-integrability) and the Cauchy–Picard theorem requires the holomorphy property of the right-hand side of the differential system written in the canonical form of Cauchy, and implies the existence, uniqueness and holomorphy property of the solution. Thus, to sum up, the Painlevé test is clearly restricted to holomorphic solutions and probes the singularity structure of the general solution of ODEs, when analytically continued into the complex domain of the independent variable. The holomorphy requirement is the sole restriction of the method and it enables us to undertake the Painlevé local singularity analysis, which can produce the most elementary results related with the possible integrability of the system. In the inconclusive case, the Painlevé test does not provide any result at all but, in the conclusive case, it definitely proves the non-integrability of the system under consideration.

The second paper of CGR and the work of LMC show that the 'mixmaster' dynamics is definitely not integrable in the Painlevé sense, since the existence of movable transcendental essential singularities (logarithmic branchings) can be proved with the help of the perturbative Painlevé test, formalized quite recently by Conte, Fordy and Pickering (hereafter CFP) [7]. One of the great advantages of such a technique is that it can display very useful information about all physical single-valued particular solutions that may happen to be present if the general solution itself exhibits movable critical singularities. Moreover, the Painlevé method can be used to test the structural stability of the chaotic solutions presumably present in the vacuum case with respect to matter fields or different dimensionality.

This paper is devoted to the study of the perfect fluid 'mixmaster' relativistic cosmological model with the help of the Painlevé technique. The plan of the paper is as follows. In section 2, we obtain the Bianchi type-IX relativistic field equations in the presence of a perfect fluid with a barotropic state equation and we define our notation. Each following section is devoted to a particular physical case. In section 3, we perform the CFP perturbative Painlevé test in the vicinity of a non-critical movable singularity, in the case of stiff-matter and radiation-dominated universes, although these two cases are treated separately. This section also briefly presents the Painlevé method. In section 4, we study the physical case of a matter-dominated cosmological model. In section 5, we end our investigations with the study of a vacuum 'mixmaster' model in the presence of a cosmological constant term. Our results are discussed in section 6.

2. The model and notation

A suitable definition of the dynamical system can be obtained very easily if one adopts the metric functions given by the usual diagonal expression of the infinitesimal length element

$$\mathrm{d}s^2 = -a(t)b(t)c(t)\,\mathrm{d}t^2 + a(t)(\omega^1)^2 + b(t)(\omega^2)^2 + c(t)(\omega^3)^2 \tag{1}$$

where the $SO(3)$-invariant differential forms which generate the homogeneous space of Bianchi type-IX are given by the canonical invariant basis

$$\omega^1 = -\sin(x^3)\,\mathrm{d}x^1 + \sin(x^1)\cos(x^3)\,\mathrm{d}x^2 \tag{2}$$

$$\omega^2 = +\cos(x^3)\,\mathrm{d}x^1 + \sin(x^1)\sin(x^3)\,\mathrm{d}x^2 \tag{3}$$

$$\omega^3 = +\cos(x^1)\,\mathrm{d}x^2 + \mathrm{d}x^3. \tag{4}$$

The field equations with a perfect (non-viscous) fluid are obtained by varying the following action with respect to the metric:

$$S \equiv S_g + S_m = \int \mathrm{d}^4x \sqrt{-{}^{(4)}g}\,({}^{(4)}R + \mathcal{L}) \tag{5}$$

with

$$\frac{1}{2}\sqrt{-{}^{(4)}g}\,T_{\alpha\beta} = \frac{\partial\left(\sqrt{-{}^{(4)}g}\,\mathcal{L}\right)}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial x^\gamma}\left(\frac{\partial\left(\sqrt{-{}^{(4)}g}\,\mathcal{L}\right)}{\partial(\partial g^{\alpha\beta}/\partial x^\gamma)}\right) \tag{6}$$

and

$$T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + p g_{\alpha\beta} \tag{7}$$

where p and ρ are the isotropic hydrodynamic pressure and the energy density associated with the perfect fluid. A state equation is given by assuming a relation (which can be supposed linear) between pressure and energy density. We will adopt the barotropic state equation

$$p = (\gamma - 1)\,\rho. \tag{8}$$

If the fluid is at rest with respect to the reference frame, the spatial components of the tensorial field equation for a perfect fluid type-IX homogeneous geometry are (the prime denotes the t-time derivative):

$$(\log a)'' + a^2 - (b-c)^2 - (2-\gamma)abc\rho = 0 \tag{9}$$

$$(\log b)'' + b^2 - (c-a)^2 - (2-\gamma)abc\rho = 0 \tag{10}$$

$$(\log c)'' + c^2 - (a-b)^2 - (2-\gamma)abc\rho = 0 \tag{11}$$

where this differential system possesses the first integral

$$\begin{aligned}&(\log a)'(\log b)' + (\log b)'(\log c)' + (\log c)'(\log a)' \\ &\qquad -a^2 - b^2 - c^2 + 2(ab + bc + ca) - 4abc\rho = 0.\end{aligned} \tag{12}$$

The energy density is governed by a first-order nonlinear differential equation which can be obtained from the conservation equations:

$$2abc\rho' + \gamma\rho(abc)' = 0. \tag{13}$$

Very often, the efficiency of the Painlevé calculations is increased if one defines the differential system in the so-called canonical form of Cauchy (first order, solved for the first derivatives). Setting the dynamical variables

$$A \equiv -\frac{1}{2}\left(\frac{b'}{b} + \frac{c'}{c}\right) \qquad B \equiv -\frac{1}{2}\left(\frac{c'}{c} + \frac{a'}{a}\right) \qquad C \equiv -\frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) \tag{14}$$

one readily writes the differential system in the adequate form

$$A' = -a(a - b - c + (2-\gamma)bc\rho) \tag{15}$$

$$B' = -b(b - c - a + (2-\gamma)ca\rho) \tag{16}$$

$$C' = -c(c - a - b + (2-\gamma)ab\rho) \tag{17}$$

$$a' = +a(A - B - C) \tag{18}$$

$$b' = +b(B - C - A) \tag{19}$$

$$c' = +c(C - A - B) \tag{20}$$

$$\rho' = +\frac{\gamma}{2}\rho\,(A + B + C) \tag{21}$$

with the purely algebraic constraint

$$A^2 + B^2 + C^2 - 2(AB + BC + CA) + a^2 + b^2 + c^2 - 2(ab + bc + ca) = -4abc\rho. \quad (22)$$

In what follows, we will not consider the barotropic parameter γ as an unspecified parameter (this would certainly be practically intractable from the singularity analysis point of view), but we will restrict ourselves to some physically interesting cases. In section 3, we will focus our attention on the $\gamma = 2$ (stiff-matter) case and on the $\gamma = 4/3$ case (for a radiation-dominated cosmological model). In section 4, we will study the $\gamma = 1$ (dust) case while in section 5, we will treat the model in the presence of a cosmological constant (i.e. the $\gamma = 0$ case, for which we obtain ρ constant).

From the Painlevé singularity analysis point of view, all these cases are indeed very different, and each case deserves (and implies) a particular treatment. This can easily be seen if one injects the first term of a Laurent series into the conservation equation (13):

$$\begin{aligned}
&2(\log\rho)' + \gamma(\log(abc))' = 0\\
&\Rightarrow 2p_\rho\chi^{-1} + \gamma(p_a + p_b + p_c)\chi^{-1} \approx 0\\
&\Rightarrow p_\rho = -\frac{\gamma}{2}(p_a + p_b + p_c)
\end{aligned}$$

where

$$a \approx \alpha_a\chi^{p_a} \qquad b \approx \alpha_b\chi^{p_b} \qquad c \approx \alpha_c\chi^{p_c} \qquad \rho \approx \alpha_\rho\chi^{p_\rho} \qquad \chi \equiv t - t_0.$$

Thus, it is obvious that the so-called 'singularity order' p_ρ cannot be integer-valued for all possible values of the γ parameter. We recall that the presence of movable algebraic branchings in the general solution of a differential system implies that it is not integrable in the Painlevé sense, but an algebraic branching does not suffice, of course, to decide about chaos.

3. The stiff-matter and radiation cases

The first step is to obtain the principal parts and singularity orders for all the families of non-critical movable singularities. Defining the so-called leading behaviours near a movable pole

$$\begin{aligned}
&a \approx \alpha_a\chi^{p_a} \qquad b \approx \alpha_b\chi^{p_b} \qquad c \approx \alpha_c\chi^{p_c} \qquad \rho \approx \alpha_\rho\chi^{p_\rho}\\
&\chi \equiv t - t_0 \qquad \boldsymbol{p} \equiv (p_a, p_b, p_c, p_\rho) \qquad \boldsymbol{\alpha} \equiv (\alpha_a, \alpha_b, \alpha_c, \alpha_\rho)
\end{aligned} \quad (23)$$

one can find all possible values of $(\boldsymbol{p}, \boldsymbol{\alpha})$ for which two or more terms in each differential equation balance each other (these are called the 'dominant' terms), while the rest can be ignored as arising at strictly higher powers of the expansion variable $\chi \equiv t - t_0$ (for a tutorial introduction to the Painlevé tests, see Conte and Ramani *et al* [8]). From the field equations (9)–(11) and (13), we thus obtain

$$0 \approx -p_a\chi^{-2} + \alpha_a^2\chi^{2p_a} - \alpha_b^2\chi^{2p_b} - \alpha_c^2\chi^{2p_c} + 2\alpha_b\alpha_c\chi^{p_b+p_c} - (2-\gamma)\alpha_a\alpha_b\alpha_c\alpha_\rho\chi^{p_a+p_b+p_c+p_\rho} \quad (24)$$

$$0 \approx -p_b\chi^{-2} + \alpha_b^2\chi^{2p_b} - \alpha_c^2\chi^{2p_c} - \alpha_a^2\chi^{2p_a} + 2\alpha_c\alpha_a\chi^{p_c+p_a} - (2-\gamma)\alpha_a\alpha_b\alpha_c\alpha_\rho\chi^{p_a+p_b+p_c+p_\rho} \quad (25)$$

$$0 \approx -p_c\chi^{-2} + \alpha_c^2\chi^{2p_c} - \alpha_a^2\chi^{2p_a} - \alpha_b^2\chi^{2p_b} + 2\alpha_a\alpha_b\chi^{p_a+p_b} - (2-\gamma)\alpha_a\alpha_b\alpha_c\alpha_\rho\chi^{p_a+p_b+p_c+p_\rho} \quad (26)$$

$$0 \approx 2p_\rho\chi^{-1} + \gamma(p_a + p_b + p_c)\chi^{-1}. \quad (27)$$

The $\gamma = 2$ (stiff-matter) case is obvious: one is led to an ordinary differential system equivalent to the one obtained in the vacuum case, and the energy density of matter is given by the relation $abc\rho = \text{constant} \equiv \mathcal{C}$. As far as the purely geometrical part of the equations is concerned, one is led to the Bianchi type-IX vacuum model which has already been investigated. There exist two interesting families given by the behaviours

$$f1 \quad : \quad a \approx \mathrm{i}\chi^{-1} \qquad b \approx b_0\chi \qquad c \approx c_0\chi \tag{28}$$

$$f2 \quad : \quad a \approx \mathrm{i}\chi^{-1} \qquad b \approx \mathrm{i}\chi^{-1} \qquad c \approx \mathrm{i}\chi^{-1}. \tag{29}$$

This is the first step of the Painlevé analysis: finding the principal parts and singularity orders for all the families of non-critical movable singularities. The second step is to compute the so-called 'Kowalevskaya resonances' (also named 'Fuchs indices' by some authors). For each resonance, one must ensure that it is indeed possible to introduce arbitrary coefficients in the Laurent expansion which is meant to be a local representation of the general solution of the system. Some preliminary conditions must also be satisfied. All these conditions will be recalled below, here we shall concentrate on the resonances themselves.

The resonances are given by the zeros of the indicial equation associated with the recurrence relation which must be solved in order to generate a Laurent series for the general solution (see this recurrence relation in equation (33) which is stated below). A rank necessary condition must be satisfied by every resonance: the multiplicity of each resonance must equal the dimensionality of the kernel associated with each particular value of the matrix needed to define the recurrence relation for the Laurent coefficients. This condition ensures that the number of arbitrary parameters introduced in the Laurent series is equal to the multiplicity of the resonance. If this condition is not satisfied, logarithmic branchings enter the expansion and the system is not integrable in the Painlevé sense. Together with the usual conditions for the absence of algebraic branchings, these are *sine qua non* necessary conditions for a differential system to be integrable in the Painlevé sense. Other necessary conditions are generated within the test itself. In the present case, one finds the resonances

$$f1 \quad : \quad r = (-1(1), 0(2), 1(2), 2(1)) \tag{30}$$

$$f2 \quad : \quad r = (-1(3), 2(3)) \tag{31}$$

where the multiplicities are given between parentheses. These two families have been tested recently by CGR and LMC. This is the third and last step of the Painlevé test: one must ensure that it is indeed possible to generate the Laurent series. The Painlevé test thus generates some necessary conditions for the differential system to possess the so-called 'Painlevé property' (hereafter PP). We recall that a differential system possesses the PP if and only if its general solution does not exhibit any movable branching.

In particular, LMC showed that the series for the first family (28) represents a locally meromorphic behaviour of the general solution (this maximal and principal family can be tested with the help of the 'standard' Kowalevskaya–Gambier Painlevé test and does not lead to any violation of the necessary conditions for a differential system to possess the PP). On the other hand, the series for the second family (29) only represents a four-parameter locally single-valued particular solution and one must perturb this Laurent series in order to check for a possible multivaluedness in the general solution (this is due to the presence of a triple negative resonance: one is compelled to perturb the Laurent series in order to extract the information associated with the two remaining negative resonances).

The CFP perturbative method checks the absence of movable logarithmic branchings at each perturbation order and for each resonance (positive, zero or negative). In the optimistic case where no violation occurred, this method then produces a doubly infinite Laurent series which is meant to be a local representation of a single-valued function. In this case, the PP (which is a global, not local, property) is definitely not proved, since the necessary conditions obtained do not suffice to prove single-valuedness. Moreover, the necessary conditions written by the Painlevé test are not a 'complete' set of conditions. On the other hand, if a violation does indeed occur, then the Painlevé test is definitely conclusive and shows that the dynamics under study is not integrable in the Painlevé sense (failing the test does suffice to prove multivaluedness). In this latter case, the local necessary conditions for the absence of logarithmic multivaluedness can be used to extract all possible physical single-valued particular solutions and to test their single-valuedness. If the conditions are definitely sufficient, these single-valued particular solutions should be written in a closed form.

Let us recall that the violation of the PP in itself does not suffice to decide about chaos. Nevertheless, the presence of an infinity of movable logarithms is generally admitted to be sufficient to ensure the existence of chaotic regimes, and this shows the pertinence of the Painlevé test.

To perform this perturbative test up to a maximal given order N, one defines the following double series (one perturbative Taylor series and one Laurent 'Kowalevskaya–Gambier' series):

$$\boldsymbol{u} = \sum_{n=0}^{N} \epsilon^n \sum_{j=nj_-}^{(n-N)j_-+j_+} \boldsymbol{u}_j^{(n)} \chi^{j+\boldsymbol{p}} \tag{32}$$
$$\boldsymbol{u} \equiv (a, b, c, \rho)$$
$$\boldsymbol{u}_j^{(n)} \equiv (a_j^{(n)}, b_j^{(n)}, c_j^{(n)}, \rho_j^{(n)})$$
$$\boldsymbol{p} \equiv (p_a, p_b, p_c, p_\rho)$$

where j_- and j_+ denote respectively the lowest and highest resonances. The strategy of the test is to try and compute all the coefficients in the Laurent series. One thus tries to solve a recurrence relation for these coefficients:

$$\forall n \geqslant 0 \qquad \forall j \geqslant nj_- \qquad (n, j) \neq (0, 0) \quad : \quad P(j)\boldsymbol{u}_j^{(n)} + \boldsymbol{Q}_j^{(n)} = 0 \tag{33}$$

where $\boldsymbol{Q}_j^{(n)}$ is a polynomial function of all the previously computed coefficients. Obviously, if the index j is indeed a resonance r, then an orthogonality condition must be satisfied for all the values of the previously computed coefficients. Such 'Fuchsian' necessary conditions are generated at each perturbation order and for each resonance. The orthogonality necessary condition is given by the relation

$$\forall n \geqslant 0 \qquad \forall r (n, r) \neq (0, 0) \quad : \quad \boldsymbol{Q}_{j=r}^{(n)} \perp \ker(\tilde{P}(j = r)) \tag{34}$$

while the rank condition mentioned above reads

$$\forall r \quad : \quad \mathrm{mult}(j = r) = \dim \ker(P(j = r)). \tag{35}$$

If conditions (35) and (34) are satisfied, then the recurrence relation (33) can indeed be solved and the correct number of arbitrary parameters can be inserted in the Laurent series: it is indeed possible to generate the Laurent series without introducing branchings of any kind (the only movable singularities are poles).

With the help of this perturbative test, LMC have shown that the Bianchi type-IX dynamics does not possess the PP in the vacuum case, exhibiting movable transcendental essential singularities in the local representation of the general solution generated by the family (29).

Moreover, these authors showed that there exists no single-valued solution to the Bianchi type-IX vacuum model other than the three known ones (i.e. the solution of Halphen for the system of Darboux [9], the axisymmetric particular solution of Taub [10] and a solution which should be a four-parameter exact and closed-form generalization of the three-parameter single-valued solution of Belinskij, Gibbons, Page and Pope (hereafter BGPP [11]), which is the general solution of the system of Euler).

It shall be stressed here that the system of Euler and the system of Darboux (both mentioned above) are particular cases of a more general sixth-order differential system which can be obtained by studying scalar-flat metrics with anti-self-dual Weyl tensor and which has been studied and integrated recently by Tod (1994) [11]. This latter system can be integrated with the help of the solution of Halphen and in terms of the sixth Painlevé transcendent. The BGPP solution and the Halphen solution then describe two particular Ricci-flat Kähler metrics.

The first result of LMC remains perfectly valid for a perfect fluid Bianchi type-IX cosmological model, since this latter model is essentially a seventh-order dynamical system which contains, as a particular case, the sixth-order system of the vacuum case, which is known not to possess the PP. On the other hand, it is quite natural to conjecture that at least some of these single-valued particular solutions may not subsist in the perfect fluid case. As we will see, some of these particular solutions are not structurally stable in the presence of such a matter field.

Indeed, the $\gamma = 2$ (stiff-matter) case is no different from the vacuum case, since the components $(1, 1)$, $(2, 2)$ and $(3, 3)$ of the tensorial field equation remain the same as in the vacuum case. The $\gamma = 4/3$ (radiation) case is much more interesting, and we shall concentrate on this case.

We shall consider the dynamical system written in the canonical form of Cauchy (equations (15) to (21)) and restrict ourselves to the family

$$\begin{aligned} f1 \quad : \quad & a \approx b \approx c \approx \mathrm{i}\chi^{-1} \qquad A \approx B \approx C \approx \chi^{-1} \qquad \rho \approx \alpha_\rho \chi^{+2} \\ & \boldsymbol{p} = (-1, -1, -1, -1, -1, -1, +2) \\ & \boldsymbol{\alpha} = (\mathrm{i}, \mathrm{i}, \mathrm{i}, 1, 1, 1, \alpha_\rho) \end{aligned} \tag{36}$$

since it does lead to conclusive results (for completeness, one should make use of every single family, since the omission of one family can easily lead to erroneous conclusions, but this is not necessary here since the family under consideration does lead to a violation). The resonances are the seven zeros of the indicial equation

$$\det(P(j)) \equiv \det \begin{pmatrix} j & 0 & 0 & -\mathrm{i} & \mathrm{i} & \mathrm{i} & 0 \\ 0 & j & 0 & \mathrm{i} & -\mathrm{i} & \mathrm{i} & 0 \\ 0 & 0 & j & \mathrm{i} & \mathrm{i} & -\mathrm{i} & 0 \\ 0 & -\mathrm{i} & -\mathrm{i} & j-1 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & -\mathrm{i} & 0 & j-1 & 0 & 0 \\ -\mathrm{i} & -\mathrm{i} & 0 & 0 & 0 & j-1 & 0 \\ 0 & 0 & 0 & -2\alpha_\rho & -2\alpha_\rho & -2\alpha_\rho & 3j \end{pmatrix} = 0$$

and we find the resonances (or 'Fuchs indices'):

$$f1 \quad : \quad r = (-1(3), 0(1), 2(3)). \tag{37}$$

A perturbative Painlevé routine, written by us in the algebraic programming language REDUCE [12], has computed the following coefficients (the coefficients for b, c, B and C can readily be written with the help of the cyclicity of the family; symbols c_i always denote arbitrary constants):

$a_0^{(0)} = \mathrm{i} \qquad A_0^{(0)} = 1 \qquad \rho_0^{(0)} = \alpha_\rho$

$a_{-1}^{(1)} = c_1 \qquad 2\mathrm{i}A_{-1}^{(1)} = c_2 + c_3 \qquad 3\mathrm{i}\rho_{-1}^{(1)} = -2\alpha_\rho(c_1 + c_2 + c_3)$

$4\mathrm{i}a_{-2}^{(2)} = 4c_1^2 - (c_2 - c_3)^2$

$4A_{-2}^{(2)} = 2c_1(c_1 - c_2 - c_3) - (c_2^2 + c_3^2)$

$9\rho_{-2}^{(2)} = -\alpha_\rho(2(c_1^2 + c_2^2 + c_3^2) + c_1c_2 + c_2c_3 + c_3c_1)$

$3a_1^{(0)} = -\alpha_\rho \qquad 3\mathrm{i}A_1^{(0)} = +\alpha_\rho \qquad 3\mathrm{i}\rho_1^{(0)} = 2\alpha_\rho^2$

$9\mathrm{i}a_0^{(1)} = \alpha_\rho(2c_1 - c_2 - c_3) \qquad A_0^{(1)} = 0 \qquad \rho_0^{(1)} = 0$

$a_2^{(0)} = c_4 \qquad 9A_2^{(0)} = 9\mathrm{i}(c_5 + c_6) - \alpha_\rho^2$

$3\rho_2^{(0)} = \alpha_\rho(2\mathrm{i}(c_4 + c_5 + c_6) - \alpha_\rho^2)$

$27a_1^{(1)} = 27\mathrm{i}(c_1c_4 - (c_2 - c_3)(c_5 - c_6)) + \alpha_\rho^2(4c_1 + 7(c_2 + c_3))$

$54A_1^{(1)} = +27(2c_1c_4 - c_1(c_5 + c_6) - c_2(c_4 + c_5) - c_3(c_4 + c_6)) + \mathrm{i}\alpha_\rho^2(16c_1 + 7(c_2 + c_3))$

$$9\rho_1^{(1)} = +4\mathrm{i}\alpha_\rho^3(c_1 + c_2 + c_3) - 4\alpha_\rho(c_1c_4 + c_2c_5 + c_3c_6) - 10\alpha_\rho(c_1(c_5 + c_6) + c_2(c_6 + c_4) + c_3(c_4 + c_5))$$

$54a_3^{(0)} = \alpha_\rho(3\mathrm{i}(11c_4 - c_5 - c_6) - \alpha_\rho^2)$

$54A_3^{(0)} = \alpha_\rho(9(c_4 - 2c_5 - 2c_6) - \mathrm{i}\alpha_\rho^2)$

$9\rho_3^{(0)} = \alpha_\rho^2(3(c_4 + c_5 + c_6) + \mathrm{i}\alpha_\rho^2).$

As it should, one arbitrary coefficient is introduced at zeroth perturbation order for the resonance 0 (this coefficient is the principal part for the energy density). Three other arbitrary coefficients enter the double expansion at first perturbation order for the resonance -1. Finally, the last three arbitrary coefficients are introduced at zeroth order for the resonance 2. To sum up, the seven arbitrary parameters are thus

$$\rho_0^{(0)} = \alpha_\rho \qquad a_{-1}^{(1)} = c_1 \qquad b_{-1}^{(1)} = c_2 \qquad c_{-1}^{(1)} = c_3$$
$$a_2^{(0)} = c_4 \qquad b_2^{(0)} = c_5 \qquad c_2^{(0)} = c_6.$$

One violation of the necessary conditions (34) occurs at second perturbation order for the resonance -1, i.e. when the step $(n, j) = (2, -1)$ is reached. A second violation occurs when $(n, j) = (1, 2)$. This means that the radiation-dominated perfect fluid Bianchi type-IX relativistic dynamics is definitely not integrable in the Painlevé sense. Of course, such a result is by no means a surprise. Indeed, the seventh-order differential system studied here can be considered as a general case which contains a sixth-order particular differential system, namely the vacuum Bianchi type-IX relativistic model. This latter model has recently been extensively studied by CGR and LMC, showing that, even in the vacuum case, this model does not possess the PP. Thus, it is obvious that our perfect fluid model cannot possess the PP. Nevertheless, the structure of the violations encountered here is quite interesting and does lead to some general results, showing the efficiency of the Painlevé strategy. The two necessary conditions (34):

$$\boldsymbol{Q}_{-1}^{(2)} \perp \ker(\tilde{P}(j = -1)) \qquad \boldsymbol{Q}_{+2}^{(1)} \perp \ker(\tilde{P}(j = +2))$$

are not satisfied under generic circumstances. In order to satisfy the first condition, one should manage to obtain

$$0 = 54\mathcal{C}_{-1,1}^{(2)} = \alpha_\rho(-7c_1^2 + 7(+c_1c_2 - c_2c_3 + c_3c_1) + 8(c_2 - c_3)^2) \tag{38}$$

$$0 = 54\mathcal{C}^{(2)}_{-1,2} = \alpha_\rho(-7c_2^2 + 7(+c_1c_2 + c_2c_3 - c_3c_1) + 8(c_3 - c_1)^2) \tag{39}$$

$$0 = 54\mathcal{C}^{(2)}_{-1,3} = \alpha_\rho(-7c_3^2 + 7(-c_1c_2 + c_2c_3 + c_3c_1) + 8(c_1 - c_2)^2). \tag{40}$$

Simultaneously, in order to satisfy the second condition, we must set

$$0 = 27\mathrm{i}\mathcal{C}^{(1)}_{+2,1} = +32\mathrm{i}\alpha_\rho c_1c_4 + 14\mathrm{i}\alpha_\rho(c_2c_6 + c_3c_5) + 2\mathrm{i}\alpha_\rho c_1(c_5 + c_6) - 16\mathrm{i}\alpha_\rho(c_2(c_4 + c_5) + c_3(c_4 + c_6)) - 2\alpha_\rho^3(2c_1 - c_2 - c_3) \tag{41}$$

$$0 = 27\mathrm{i}\mathcal{C}^{(1)}_{+2,2} = +32\mathrm{i}\alpha_\rho c_2c_5 + 14\mathrm{i}\alpha_\rho(c_3c_4 + c_1c_6) + 2\mathrm{i}\alpha_\rho c_2(c_6 + c_4) - 16\mathrm{i}\alpha_\rho(c_3(c_5 + c_6) + c_1(c_5 + c_4)) - 2\alpha_\rho^3(2c_2 - c_3 - c_1) \tag{42}$$

$$0 = 27\mathrm{i}\mathcal{C}^{(1)}_{+2,3} = +32\mathrm{i}\alpha_\rho c_3c_6 + 14\mathrm{i}\alpha_\rho(c_1c_5 + c_2c_4) + 2\mathrm{i}\alpha_\rho c_3(c_4 + c_5) - 16\mathrm{i}\alpha_\rho(c_1(c_6 + c_4) + c_2(c_6 + c_5)) - 2\alpha_\rho^3(2c_3 - c_1 - c_2). \tag{43}$$

Imposing ‘by hand’ some relations between the seven arbitrary coefficients, one can indeed satisfy these two vectorial conditions, which are violated in the general case. This technique produces all the physically interesting single-valued particular solutions associated with a given ‘unstable’ (in the Painlevé sense) differential system. It can lead to some exact, closed-form, solutions. In our case, the nonlinear algebraic system

$$(\mathcal{C}^{(2)}_{-1} = 0, \mathcal{C}^{(1)}_{+2} = 0)$$

is only identically satisfied under the following circumstances (obtained with the help of the GROEBNER package [13] in REDUCE):

$$(\alpha_\rho = 0) \qquad (c_1 = c_2 = c_3).$$

The perturbative method seems to introduce ‘another’ arbitrary coefficient at first perturbation order for the resonance -1. Since the formal solution does not, of course, depend on eight independent parameters, one can freely impose one scalar relation between the four quantities related to the triple Fuchs index -1, i.e. the arbitrary parameters (t_0, c_1, c_2, c_3). Thus, without loss of generality, we rewrite the two conditions

$$(\alpha_\rho = 0) \tag{44}$$

$$(c_1 = c_2 = c_3 \equiv 0). \tag{45}$$

In the first case, we are brought back to the vacuum model. In the second case, the perturbative test reduces to the Kowalevskaya–Gambier test. These two possibilities will be discussed in section 6.

4. The dust case

In the $\gamma = 1$ (dust) case, the conservation equation produces

$$\rho = \mathcal{C}\frac{1}{\sqrt{abc}} \quad : \quad \mathcal{C} = \text{constant}. \tag{46}$$

The best way to consider the differential system for this particular (dust) case is to write down the Einstein equations

$$G_{00} = T_{00} = abc\rho \qquad G_{ii} = T_{ii} = 0 \quad : \quad \forall i = 1, 2, 3 \tag{47}$$

for a type-IX spatially homogeneous geometry, producing the system

$$(\log a)'' - \tfrac{1}{4}((\log a)'(\log b)' + (\log b)'(\log c)' + (\log c)'(\log a)') + \tfrac{1}{4}(5a^2 - 3b^2 - 3c^2) - \tfrac{1}{2}(ab + ac - 3bc) = 0 \tag{48}$$

$$(\log b)'' - \tfrac{1}{4}((\log a)'(\log b)' + (\log b)'(\log c)' + (\log c)'(\log a)') + \tfrac{1}{4}(5b^2 - 3c^2 - 3a^2) - \tfrac{1}{2}(bc + ba - 3ca) = 0 \tag{49}$$

$$(\log c)'' - \tfrac{1}{4}((\log a)'(\log b)' + (\log b)'(\log c)' + (\log c)'(\log a)') + \tfrac{1}{4}(5c^2 - 3a^2 - 3b^2) - \tfrac{1}{2}(ca + cb - 3ab) = 0. \tag{50}$$

We obtain the two following singularity families:

$$f1 \quad : \quad \boldsymbol{p} = (-1, +1, +1) \qquad \boldsymbol{\alpha} = (\pm\mathrm{i}, b_0, c_0) \tag{51}$$

$$f2 \quad : \quad \boldsymbol{p} = (-1, -1, -1) \qquad \boldsymbol{\alpha} = (\pm\mathrm{i}, \pm\mathrm{i}, \pm\mathrm{i}) \tag{52}$$

for which the resonances ('Fuchs indices') are

$$f1 \quad : \quad r = (-1(1), 0(2), 1(2), \tfrac{5}{2}(1)) \tag{53}$$

$$f2 \quad : \quad r = (-1(3), \tfrac{1}{2}(1), 2(2)). \tag{54}$$

Each of these families describes one six-parameter, bivalued, local solution (this bivaluation is due to the presence of half-integer resonances). Again, the Painlevé method can be used to check the absence of logarithmic multivaluation. The first family (51) can be used to generate a local representation whose coefficients can be computed (without any violation of the necessary conditions) with the help of the standard Kowalevskaya test. On the other hand, the second family (52) can be used to generate another local representation whose series must be perturbed in order to extract some information correlated with the negative resonances. We have performed the CFP perturbative test and we have obtained a violation of the conditions for a system to possess the PP. This violation occurs at third perturbation order for the resonance -1. We have computed the coefficients

$$a_0^{(0)} = \mathrm{i} \qquad b_0^{(0)} = \mathrm{i} \qquad c_0^{(0)} = \mathrm{i}$$

$$a_{-1}^{(1)} = c_1 \qquad b_{-1}^{(1)} = c_2 \qquad c_{-1}^{(1)} = c_3$$

$$4\mathrm{i}a_{-2}^{(2)} = 4c_1^2 - (c_2 - c_3)^2$$

$$4\mathrm{i}b_{-2}^{(2)} = 4c_2^2 - (c_3 - c_1)^2$$

$$4\mathrm{i}c_{-2}^{(2)} = 4c_3^2 - (c_1 - c_2)^2$$

$$4a_{-3}^{(3)} = -4c_1^3 + c_1(c_2 - c_3)^2 + c_2^3 + c_3^3 - c_2c_3(c_2 + c_3)$$

$$4b_{-3}^{(3)} = -4c_2^3 + c_2(c_3 - c_1)^2 + c_3^3 + c_1^3 - c_3c_1(c_3 + c_1)$$

$$4c_{-3}^{(3)} = -4c_3^3 + c_3(c_1 - c_2)^2 + c_1^3 + c_2^3 - c_1c_2(c_1 + c_2)$$

$$a_1^{(0)} = 0 \qquad b_1^{(0)} = 0 \qquad c_1^{(0)} = 0$$

$$a_0^{(1)} = 0 \qquad b_0^{(1)} = 0 \qquad c_0^{(1)} = 0$$

$$a_{-1}^{(2)} = 0 \qquad b_{-1}^{(2)} = 0 \qquad c_{-1}^{(2)} = 0$$

$$a_{-2}^{(3)} = 0 \qquad b_{-2}^{(3)} = 0 \qquad c_{-2}^{(3)} = 0$$

$$a_2^{(0)} = c_4 \qquad b_2^{(0)} = c_5 \qquad c_2^{(0)} = -(c_4 + c_5)$$

$$\mathrm{i}a_1^{(1)} = 2c_5(c_2 - c_3) - c_4(c_1 - c_2 + c_3)$$

$$\mathrm{i}b_1^{(1)} = 2c_4(c_1 - c_3) + c_5(c_1 - c_2 - c_3)$$

$$\mathrm{i}c_1^{(1)} = c_4(c_1 - c_2 + c_3) - c_5(c_1 - c_2 - c_3)$$
$$2a_0^{(2)} = +c_1(c_4 + 2c_5)(c_2 - c_3) + 9c_2c_3c_4 - c_2^2(5c_4 + c_5) - c_3^2(4c_4 - c_5)$$
$$2b_0^{(2)} = +c_2(c_5 + 2c_4)(c_1 - c_3) + 9c_1c_3c_5 - c_1^2(c_4 + 5c_5) - c_3^2(4c_5 - c_4)$$
$$2c_0^{(2)} = +c_1^2(4c_4 + 5c_5) + c_2^2(5c_4 + 4c_5) - 9c_1c_2(c_4 + c_5) + c_3(c_1 - c_2)(c_4 - c_5)$$
$$a_3^{(0)} = 0 \qquad b_3^{(0)} = 0 \qquad c_3^{(0)} = 0$$
$$a_2^{(1)} = 0 \qquad b_2^{(1)} = 0 \qquad c_2^{(1)} = 0$$
$$a_1^{(2)} = 0 \qquad b_1^{(2)} = 0 \qquad c_1^{(2)} = 0$$
$$5\mathrm{i}a_4^{(0)} = 3c_4^2 - 2c_4c_5 - 2c_5^2$$
$$5\mathrm{i}b_4^{(0)} = 3c_5^2 - 2c_4c_5 - 2c_4^2$$
$$5\mathrm{i}c_4^{(0)} = 3c_4^2 + 8c_4c_5 + 3c_5^2$$
$$5a_3^{(1)} = +14c_1((c_4 + c_5)^2 - c_4c_5) - 5c_2(2(c_4 + c_5)^2 + c_4c_5) + 5c_3(c_4^2 + c_4c_5 - 2c_5^2)$$
$$5b_3^{(1)} = +14c_2((c_4 + c_5)^2 - c_4c_5) - 5c_1(2(c_4 + c_5)^2 + c_4c_5) + 5c_3(c_5^2 + c_4c_5 - 2c_4^2)$$
$$5c_3^{(1)} = +5c_1(c_4^2 + c_4c_5 - 2c_5^2) + 5c_2(c_5^2 + c_4c_5 - 2c_4^2) + 14c_3((c_4 + c_5)^2 - c_4c_5).$$

The five arbitrary parameters are thus

$$a_{-1}^{(1)} = c_1 \qquad b_{-1}^{(1)} = c_2 \qquad c_{-1}^{(1)} = c_3 \qquad a_2^{(0)} = c_4 \qquad b_2^{(0)} = c_5.$$

The $(n, j) = (3, -1)$ Fuchsian necessary condition (34) is violated in the general case. This orthogonality condition implies

$$0 = 3^{-1}\mathcal{C}_{-1,1}^{(3)} = +c_1^2(c_2 - c_3)(c_4 + 2c_5) + 12c_1c_2c_3c_4 - c_1(c_2^2(7c_4 + 2c_5) + c_3^2(5c_4 - 2c_5)) + c_2^3(4c_4 + 2c_5) - c_2^2c_3(5c_4 + 4c_5) + c_3^3(2c_4 - 2c_5) - c_3^2c_2(c_4 - 4c_5) \tag{55}$$

$$0 = 3^{-1}\mathcal{C}_{-1,2}^{(3)} = -c_2^2(c_3 - c_1)(c_5 + 2c_4) + 12c_1c_2c_3c_5 - c_2(c_1^2(2c_4 + 7c_5) + c_3^2(5c_5 - 2c_4)) + c_1^3(2c_4 + 4c_5) - c_1^2c_3(4c_4 + 5c_5) + c_3^3(2c_5 - 2c_4) - c_3^2c_1(c_5 - 4c_4) \tag{56}$$

$$0 = 3^{-1}\mathcal{C}_{-1,3}^{(3)} = -2c_1^3(c_4 + 2c_5) - 2c_2^3(2c_4 + c_5) - 12c_1c_2c_3(c_4 + c_5) + c_1^2c_2(c_4 + 5c_5) + c_1c_2^2(5c_4 + c_5) + c_1^2c_3(5c_4 + 7c_5) + c_1c_3^2(c_4 - c_5) + c_2^2c_3(7c_4 + 5c_5) + c_2c_3^2(c_5 - c_4). \tag{57}$$

This vectorial condition generates a nonlinear algebraic system whose five solutions can be written in the form

$$(c_2 = c_1, c_4 - c_5 = 0)$$
$$(c_3 = c_1, 2c_4 + c_5 = 0)$$
$$(c_3 = c_2, 2c_5 + c_4 = 0)$$
$$(c_4 = 0, c_5 = 0)$$
$$(c_1 = c_2 = c_3)$$

or, equivalently, in the more transparent form

$$(a_{-1}^{(1)} = b_{-1}^{(1)}, a_2^{(0)} = b_2^{(0)}) \tag{58}$$
$$(a_{-1}^{(1)} = c_{-1}^{(1)}, a_2^{(0)} = c_2^{(0)}) \tag{59}$$
$$(b_{-1}^{(1)} = c_{-1}^{(1)}, b_2^{(0)} = c_2^{(0)}) \tag{60}$$
$$(a_2^{(0)} = b_2^{(0)} = 0) \Rightarrow (c_2^{(0)} = 0) \tag{61}$$
$$(a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)}). \tag{62}$$

Another violation occurs at second perturbation order for the resonance 2. We shall not give this violation explicitly here, since this $(n, j) = (2, 2)$ condition does not refine the $(n, j) = (3, -1)$ condition. Indeed, the $(n, j) = (2, 2)$ necessary condition, which is violated under generic circumstances, can be satisfied if one imposes one of the five constraints

$$
\begin{aligned}
&(c_2 = c_1, c_4 - c_5 = 0)\\
&(c_3 = c_1, 2c_4 + c_5 = 0)\\
&(c_3 = c_2, 2c_5 + c_4 = 0)\\
&(c_4(c_1 - c_2) = c_5(c_2 - c_3), c_4(c_2 - c_3) = c_5(c_3 - c_1), (c_4 + c_5)^2 = c_4 c_5)\\
&(c_2 = c_1, c_3 = c_1).
\end{aligned}
$$

Thus, it is obvious that the $(n, j) = (2, 2)$ condition does not refine the $(n, j) = (3, -1)$ condition. Nevertheless, the $(n, j) = (5, -1)$ condition must also be satisfied, and this latter orthogonality condition is indeed crucial as far as the 'axisymmetric' solutions are concerned. If one imposes the constraint (61) or the constraint (62), then the $(n, j) = (5, -1)$ condition is identically satisfied. This is no longer true in the axisymmetric case: the constraints (58)–(60) do not suffice to ensure single-valuedness under generic circumstances. For example, if one imposes the constraint (58), then the $(n, j) = (5, -1)$ condition is identically satisfied if and only if one manages to set the three following quantities to zero:

$$
\begin{aligned}
&(20\,115 c_1^5 + 43\,021 c_1^4 c_3 + 74\,326 c_1^3 c_3^2 + 1650 c_1^2 c_3^3 - 26\,297 c_1 c_3^4 - 14\,063 c_3^5) c_4^2\\
&(4323 c_1^5 - 9619 c_1^4 c_3 + 21\,686 c_1^3 c_3^2 + 1650 c_1^2 c_3^3 + 23 c_1 c_3^4 - 3535 c_3^5) c_4^2\\
&(5973 c_1^5 - 6577 c_1^4 c_3 + 32\,858 c_1^3 c_3^2 - 5850 c_1^2 c_3^3 - 14\,911 c_1 c_3^4 - 10\,981 c_3^5) c_4^2.
\end{aligned}
$$

This can only be achieved if $c_4 = 0$ or if $c_1 = c_3 = 0$. Thus, the constraint (58) reduces to one of the two constraints

$$(c_4 = c_5 = 0, c_2 = c_1) \qquad \text{and} \qquad (c_1 = c_2 = c_3 = 0, c_5 = c_4).$$

The case of constraints (59) and (60) is handled in the same way. This shows that the axisymmetric solutions become subsets of cases (61) and (62). To sum up, in the dust case, the Fuchsian necessary conditions (34) may thus only be satisfied if one imposes one of the constraints (61) and (62). These results will be discussed in section 6.

5. The cosmological constant case

In the $\gamma = 0$ case, the conservation equation produces $\rho = +\Lambda$, where Λ is precisely the cosmological constant. The field equations $R_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0$ for a type-IX spatially homogeneous geometry read:

$$(\log a)'' + a^2 - (b - c)^2 - 2\Lambda abc = 0 \tag{63}$$

$$(\log b)'' + b^2 - (c - a)^2 - 2\Lambda abc = 0 \tag{64}$$

$$(\log c)'' + c^2 - (a - b)^2 - 2\Lambda abc = 0. \tag{65}$$

The vacuum relativistic Bianchi type-IX family $\boldsymbol{p} = (-1, -1, -1)$ does not survive here. Moreover, all the families $\boldsymbol{p} = (p_a, p_b, p_c)$ with $p_a = p_b = p_c$ immediately lead to algebraic branchings: $\boldsymbol{p} = (-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3})$. On the other hand, the vacuum relativistic Bianchi type-IX family

$$f1 \quad : \quad \boldsymbol{p} = (-1, +1, +1) \qquad \boldsymbol{\alpha} = (\pm \mathrm{i}, b_0, c_0) \tag{66}$$

does indeed survive here. All the families $\boldsymbol{p} = (-1, -1, p_c \geqslant 1)$ do not exist, but at the 'extreme' limit $p_c = 0$, we obtain the family

$$f2 \quad : \quad \boldsymbol{p} = (-1, -1, 0) \qquad \boldsymbol{\alpha} = \left(\pm\frac{\mathrm{i}}{2}, \mp\frac{\mathrm{i}}{2}, \frac{2}{\Lambda}\right). \tag{67}$$

The maximal and principal family (66) can be tested with the help of the classical Kowalevskaya test (i.e. the perturbative test reduced to its zeroth perturbation order). The resonances (and their multiplicities) are

$$f1 \quad : \quad r = (-1(1), 0(2), 1(2), 2(1)) \tag{68}$$

and, without any violation of the necessary conditions, we have computed the following Laurent series coefficients:

$$\begin{array}{lll}
a_0^{(0)} = \mathrm{i} & b_0^{(0)} = b_0 & c_0^{(0)} = c_0 \\
a_1^{(0)} = 0 & b_1^{(0)} = c_1 & c_1^{(0)} = c_2 \\
a_2^{(0)} = c_3 & & \\
\multicolumn{3}{l}{2b_0 b_2^{(0)} = c_1^2 - 2\mathrm{i}b_0^2(c_0 - c_3)} \\
\multicolumn{3}{l}{2c_0 c_2^{(0)} = c_2^2 - 2\mathrm{i}c_0^2(b_0 - c_3)} \\
\multicolumn{3}{l}{2a_3^{(0)} = -\Lambda b_0 c_0} \\
\multicolumn{3}{l}{6\mathrm{i}b_0^2 b_3^{(0)} = +\mathrm{i}c_1^3 - \Lambda b_0^4 c_0 + 2b_0^3 c_2 + 6b_0^2 c_1(c_0 - c_3)} \\
\multicolumn{3}{l}{6\mathrm{i}c_0^2 c_3^{(0)} = +\mathrm{i}c_2^3 - \Lambda b_0 c_0^4 + 2c_0^3 c_1 + 6c_0^2 c_2(b_0 - c_3)}
\end{array}$$

where the $(n, j) = (0, 3)$ coefficients are given (even though 3 is not a resonance) because the cosmological constant Λ enters the expansion when this step is reached. The six arbitrary parameters are thus

$$(t_0, b_0, c_0, c_1, c_2, c_3)$$

and we obtain the truncated Laurent series

$$a \approx \mathrm{i}\chi^{-1} + c_3\chi \tag{69}$$

$$b \approx b_0\chi + c_1\chi^2 + \left(\mathrm{i}b_0c_3 + \frac{c_1^2}{2b_0} - \mathrm{i}b_0c_0\right)\chi^3 \tag{70}$$

$$c \approx c_0\chi + c_2\chi^2 + \left(\mathrm{i}c_0c_3 + \frac{c_2^2}{2c_0} - \mathrm{i}b_0c_0\right)\chi^3. \tag{71}$$

As for the non-maximal family (67), we only obtain four resonances, showing that the series generated by this family cannot locally represent the general solution of the system, which is the six-parameter solution. In the optimistic case where no necessary condition is violated (which is not the case here), this family (67) can only represent a four-parameter locally single-valued particular solution and tells nothing about some possible multivaluedness in the general solution. Let us recall once again that multivaluedness has been proved for the $\Lambda = 0$ case, which is known not to possess the PP [6]. In fact, the family (67) is indeed interesting, since there do exist some real irrational resonances, implying movable branchings:

$$f2 \quad : \quad r = \left(-1(1), \frac{1-\sqrt{5}}{2}(1), \frac{1+\sqrt{5}}{2}(1), +2(1)\right). \tag{72}$$

These results will be discussed in section 6.

6. Discussion

The $\gamma = 2$ (stiff-matter) case is obvious: the results of LMC [6] for the vacuum case remain perfectly valid here, and the energy density is given by the relation $\rho \propto a^{-1}b^{-1}c^{-1}$. We recall that, in the vacuum case, the $(n, j) = (3, -1)$ violated condition has the five solutions

$$(a_{-1}^{(1)} = b_{-1}^{(1)}, a_2^{(0)} = b_2^{(0)}) \tag{73}$$

$$(a_{-1}^{(1)} = c_{-1}^{(1)}, a_2^{(0)} = c_2^{(0)}) \tag{74}$$

$$(b_{-1}^{(1)} = c_{-1}^{(1)}, b_2^{(0)} = c_2^{(0)}) \tag{75}$$

$$(a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)}) \tag{76}$$

$$(a_2^{(0)} = b_2^{(0)} = c_2^{(0)}). \tag{77}$$

The first three possibilities represent the four-parameter single-valued axisymmetric particular solution discovered by Taub. The fourth condition corresponds to another four-parameter single-valued particular solution whose exact and closed-form expression is yet unknown and which should be a four-parameter generalization of the three-parameter solution discovered by BGPP. The last possibility does not suffice to ensure single-valuedness for the particular solution, since it is restricted by the $(n, j) = (5, -1)$ violated condition. This latter condition has the four solutions

$$(a_{-1}^{(1)} = b_{-1}^{(1)}, a_2^{(0)} = b_2^{(0)} = c_2^{(0)}) \tag{78}$$

$$(a_{-1}^{(1)} = c_{-1}^{(1)}, a_2^{(0)} = c_2^{(0)} = b_2^{(0)}) \tag{79}$$

$$(b_{-1}^{(1)} = c_{-1}^{(1)}, b_2^{(0)} = c_2^{(0)} = a_2^{(0)}) \tag{80}$$

$$(a_2^{(0)} = b_2^{(0)} = c_2^{(0)} \equiv 0). \tag{81}$$

The first three possibilities obviously correspond to a three-parameter subset of the more general four-parameter axisymmetric solution which is already underlined. The last possibility corresponds to the three-parameter single-valued solution of Halphen for the system of Darboux.

We recall that, in the $\gamma = 2$ case, an exact and closed-form single-valued particular solution has been obtained by Barrow [14]. This solution generalizes the vacuum axisymmetric Taub universe.

The $\gamma = 4/3$ (radiation) case is more interesting. If we impose our first constraint (44) $\alpha_\rho = \rho_0^{(0)} \equiv 0$, then we are once again brought back to the vacuum particular case (all the Laurent series coefficients $\rho_j^{(n)}$ for the energy density vanish if the principal part $\alpha_\rho = \rho_0^{(0)}$ vanishes itself). This possibility thus reduces to the already studied case of a vacuum cosmological model, and the condition $\alpha_\rho = \rho_0^{(0)} \equiv 0$ does not suffice to ensure single-valuedness, according to the LMC and CGR results.

On the other hand, if we impose our second constraint (45) $c_1 = c_2 = c_3 \equiv 0$, then we obtain precisely one of the cases (76) discovered by LMC for the vacuum case (with an adequate gauge choice):

$$(a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)} \equiv 0). \tag{82}$$

This is the only possibility of disentangling oneself from the violations encountered if one imposes the presence of matter. Our first result is thus the fact that there exists at most only one four-parameter single-valued particular solution for the radiation-dominated perfect fluid Bianchi type-IX relativistic cosmological model. This means that all the other solutions, including the four-parameter axisymmetric solution discovered by Taub in the vacuum case, cannot subsist in the presence of this radiation-dominated perfect fluid. Our second result is the fact that the only possibility to satisfy the $(n, j) = (2, -1)$ and $(n, j) = (1, +2)$ conditions does indeed suffice to satisfy also the conditions which are generated in the vacuum case. This means that the four-parameter single-valued particular solution corresponding to the constraint $a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)} \equiv 0$ does indeed survive in the presence of the radiation-dominated perfect fluid (compare our constraint (45) with the LMC 'vacuum' constraint (76)).

For the $\gamma = 1$ (dust) case, we have found the maximal families (51) and (52), each one of them exhibiting one real rational resonance. This means that each one of these families can be used to generate one six-parameter, bivalued, local solution. However, in the case of our second family (52), some necessary conditions could not be satisfied under generic circumstances (even in the case of the presence of non-integer resonances, it is always recommended to generate the orthogonality necessary conditions, since this can provide valuable information about some possible particular solutions). We have shown that the constraint (with an adequate gauge choice)

$$(a_{-1}^{(1)} = b_{-1}^{(1)} = c_{-1}^{(1)} \equiv 0) \tag{83}$$

does indeed locally represent a single-valued particular solution whose three arbitrary parameters are t_0, c_4 and c_5. Moreover, we have shown that the particular solution described by the constraint

$$(a_2^{(0)} = b_2^{(0)} = 0) \Rightarrow (c_2^{(0)} = 0) \tag{84}$$

does seem to survive in the presence of dust. In the vacuum case, this latter constraint only appears at fifth perturbation order, but in the dust case, it appears sooner, at third order. However, in the dust case, our main result is the fact that there cannot exist any exact and closed-form axisymmetric solution. This can only be proved at fifth perturbation order and this is due to the presence of logarithmic movable branchings.

Finally, for the $\gamma = 0$ (cosmological constant) case, we have shown that the vacuum relativistic Bianchi type-IX family (66) survives and does not lead to any violation of the necessary conditions for a differential system to possess the PP. On the other hand, we have found one non-maximal family (67) exhibiting some real irrational resonances, implying movable branchings. We recall that the $\Lambda = 0$ case is not integrable in the Painlevé sense. Nevertheless, imposing $\Lambda \neq 0$, we have not been able to detect any movable logarithmic branching in the local representation of the general solution. We recall that if no violation occurs, the Painlevé test does not suffice to prove the integrability of a differential system.

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References

[1] Lifchitz E M and Khalatnikov I M 1963 *Adv. Phys.* **12** 185

Belinskij V A and Khalatnikov I M 1969 *Sov. Phys.–JETP* **29** 911
Lifchitz E M and Khalatnikov I M 1970 *JETP Lett.* **11** 123
Belinskij V A, Khalatnikov I M and Lifchitz E M 1970 *Adv. Phys.* **19** 525
Belinskij V A and Khalatnikov I M 1971 *Sov. Phys.–JETP* **32** 169
Lifchitz E M, Lifchitz I M and Khalatnikov I M 1971 *Sov. Phys.–JETP* **32** 173
Belinskij V A, Lifchitz E M and Khalatnikov I M 1971 *Sov. Phys.–Usp.* **13** 745; 1971 *Sov. Phys.–JETP* **33** 1061; 1972 *Sov. Phys.–JETP* **35** 838
Belinskij V A and Khalatnikov I M 1973 *Sov. Phys.–JETP* **36** 591
Belinskij V A, Khalatnikov I M and Lifchitz E M 1982 *Adv. Phys.* **31** 639
Khalatnikov I M, Lifchitz E M, Khanin K M, Shchur L N and Sinai Ya G 1985 *J. Stat. Phys.* **38** 97
Kirillov A A and Kochnev A A 1987 *JETP Lett.* **46** 435
Belinskij V A 1992 *JETP Lett.* **56** 421
Ivashchuk V D, Mel'nikov V N and Kirillov A A 1994 *JETP Lett.* **60** 235
Kirillov A A 1994 *Int. J. Mod. Phys.* D **3** 431

[2] Barrow J D 1981 *Phys. Rev. Lett.* **46** 963; 1982 *Gen. Rel. Grav.* **14** 523
Chernoff D F and Barrow J D 1983 *Phys. Rev. Lett.* **50** 134
Barrow J D 1984 Chaotic behaviour and the Einstein equations *Classical General Relativity* ed W Bonnor, J Islam and M A H MacCallum (Cambridge: Cambridge University Press) p 25

[3] Misner C W 1969 *Phys. Rev. Lett.* **22** 1071

[4] Zardecki A 1983 *Phys. Rev.* D **28** 1235
Burd A B, Buric N and Ellis G F R 1990 *Gen. Rel. Grav.* **22** 349
Hobill D, Bernstein D, Welge M and Simkins D 1991 *Class. Quantum Grav.* **8** 1155
Szydlowski M and Biesiada M 1991 *Phys. Rev.* D **44** 2369
Szydlowski M and Szczesny J 1994 *Phys. Rev.* D **50** 819
Francisco G and Matsas G E A 1988 *Gen. Rel. Grav.* **20** 1047
Ferraz K and Francisco G 1992 *Phys. Rev.* D **45** 1158
Berger B K 1990 *Class. Quantum Grav.* **7** 203; 1991 *Gen. Rel. Grav.* **23** 1385; 1993 *Phys. Rev.* D **47** 3222
Demaret J and De Rop Y 1993 *Phys. Lett.* **299B** 223

[5] Bogoyavlenskij O I 1985 Methods in the qualitative theory of dynamical systems *Astrophysics and Gas Dynamics* (Berlin: Springer)
Wainwright J and Hsu L 1989 *Class. Quantum Grav.* **6** 1409
Ma P K H and Wainwright J 1994 A dynamical systems approach to the oscillatory singularity in Bianchi cosmologies *Deterministic Chaos in General Relativity* ed D Hobill, A Burd and A Coley (New York: Plenum) p 449

[6] Contopoulos G, Grammaticos B and Ramani A 1993 *J. Phys. A: Math. Gen.* **26** 5795; 1994 *J. Phys. A: Math. Gen.* **27** 5357
Latifi A, Musette M and Conte R 1994 *Phys. Lett.* **194A** 83

[7] Fordy A P and Pickering A 1991 *Phys. Lett.* **160A** 347
Conte R, Fordy A P and Pickering A 1993 *Physica* D **69** 33
Conte R 1994 Singularities of differential equations and integrability *Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Non-Linear Waves* ed D Benest and C Froeschlé (Gif-sur-Yvette: Editions Frontières) p 49

[8] Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715; 1006
Ramani A, Grammaticos B and Bountis T 1989 *Phys. Rep.* **180** 159
Conte R 1994 Singularities of differential equations and integrability *Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Non-Linear Waves* ed D Benest and C Froeschlé (Gif-sur-Yvette: Editions Frontières) p 49

[9] Darboux G 1878 *Ann. Sci. Ecole Normale Supérieure* **7** 101
Halphen G-H 1881 *C. R. Acad. Sci., Paris* **92** 1101 (reprinted in *Oeuvres: Tome 2* (Paris: Gauthier-Villars 1918) p 475); 1886 *Traité des Fonctions Elliptiques et de leurs Applications. Première Partie, Théorie des Fonctions Elliptiques et de leurs Développements en Série* (Paris: Gauthier-Villars); 1888 *Traité des Fonctions Elliptiques et de leurs Applications. Deuxième Partie, Applications à la Mécanique, à la Physique, à la Géodésie, à la Géométrie et au Calcul Intégral* (Paris: Gauthier-Villars)

[10] Taub A H 1951 *Ann. Math.* **53** 472

[11] Belinskij V A, Gibbons G W, Page D N and Pope C N 1978 *Phys. Lett.* **76B** 433
Atiyah M F and Hitchin N J 1985 *Phys. Lett.* **107A** 21
Pedersen H and Poon Y S 1990 *Class. Quantum Grav.* **7** 1707
Tod K P 1991 *Class. Quantum Grav.* **8** 1049

Dancer A S and Strachan I A B 1994 *Math. Proc. Camb. Phil. Soc.* **115** 513
Strachan I A B 1994 *Theor. Math. Phys.* **99** 790
Tod K P 1994 *Phys. Lett.* **190A** 221

[12] Hearn A C 1993 *REDUCE: User's Manual Version 3.5* (Santa Monica, CA: The Rand Corporation) RAND Publication CP78

[13] Melenk H, Möller H M and Neun W 1990 *GROEBNER: A Package for Calculating Groebner Basis* (Berlin: Konrad-Zuse-Zentrum für Informationstechnik)
Buchberger B 1985 Groebner basis: an algorithmic method in polynomial ideal theory *Progress, Directions and Open Problems in Multidimensional Systems Theory* ed N K Bose (Dordrecht: Reidel) p 184

[14] Barrow J D 1978 *Nature* **272** 211